More is Not Always Better: the Case of Counter-Terrorism Security*

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Abstract

Can counter-terrorism security be counter-productive? We argue that it can be when the at-risk population acts strategically. We model a two-stage game where the government first chooses the defensive security level for a public place. The second stage is a simultaneous-move game with terrorist choosing terror effort and members of the population deciding whether or not to attend the public place. Our key measure of the efficiency of the counter-terrorism security is the expected number of casualties. Under very standard and general assumptions, we show that it is possible that more security leads to an increase in that number. This is because increasing security both discourages and encourages the terrorist. On the one hand, more security makes a successful terror attack less likely (discouragement). On the other hand, more security motivates more people to attend the public place which makes the attack more valuable to the terrorist (encouragement).

Keywords: terrorism, counter-terrorism security

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1 Introduction

According to the Oxford English Dictionary, terrorism is "the unlawful use of violence and intimidation, especially against civilians, in the pursuit of political aims."[1] Because of its violent nature and targeting civilian population, the fear of terrorism is among the top worries of the modern world. In June 2017, 42% of Americans were worried about being a victim of terrorism and 66% of them consider the possibility of future terror attacks in the U.S. as a great or fair concern.[2] In 2016, 79% of American respondents pointed at international terrorism as a critical threat to the United States making it number one concern in the nation (development on nuclear weapons by Iran was the second highest with 75%).[3]

Because the fear of terrorism is widespread and significantly affects people's behavior, it is of utmost importance to incorporate it into the studies of terrorism. The impact on behavior can be seen in a June 2017 poll which shows that "as a result of the events relating to terrorism in recent years" 32% American indicated that they are less willing to fly airplanes, 26% were less willing to go to skyscrapers, 38% less willing to attend an event with thousands of people, and 46% less willing to travel abroad.[4]

Activities like going to a public place (e.g., 2008 Mumbai attacks), attending a public event (e.g., 2017 Ariana Grande concert in Manchester), or using public transportation (e.g., 2004 Madrid train bombing) are associated with the risk of being targeted by terrorists. Whatever is "public" is a potential target of a terror attack whose objective is to inflict a maximal loss of life. When deciding whether or not to be part of a "public" event and go where expected crowd is to be, people take the risk of terror attack into account and might opt out.

In order to address the fear of terrorism and prevent terror attacks, governments around the world

[1] Sandler (2015) provides a detailed discussion on the definitions of terrorism used in the literature.

[2] Source: Gallup poll, `http://news.gallup.com/poll/4909/terrorism-united-states.aspx`. Retrieved on January 12, 2018.

[3] Source: Gallup poll, `http://news.gallup.com/poll/189161/americans-cite-cyberterrorism-among-top-three-threats.aspx`. Retrieved on January 12, 2018.

[4] Source: Gallup poll, `http://news.gallup.com/poll/4909/terrorism-united-states.aspx`. Retrieved on January 12, 2018.

spend large amounts of taxpayer money on defensive counter-terrorism security.[5] For example, in USA, that amount is around $100 billion per year (Mueller and Stewart (2014)). It is only natural to ask about the effectiveness of counter-terrorism security.

Suppose that the counter-terrorism security is costless. Is more security always better?

Instinctively, the answer is "yes." After all, more security makes a public place[6] safer. However, as we explore in this paper, the problem is more complex than what instinct would indicate and it is not unlikely that the answer is "no."

We analyze the counter-terrorism security in the context of saving lives. In order to address our question, we model a strategic interaction between the government, terrorist, and population whose members the government wants to protect against the terror attack. The government acts first and provides security that works to directly reduce the effectiveness of the terror attack in the subsequent period. In the next period, both terrorist and population members act simultaneously.

In line with mainstream research, we assume that terrorists are rational.[7] They choose the level of terror effort which directly increases the probability of terror attack being successful. The effort level is chosen to maximize the terrorists' objective function which is the expected impact of terror (measured by the number of casualties) less the cost of terror effort. The attack targets a public place. The more populated the public place is, the more attractive it is for the terrorist. Hence, the terrorist takes into account how many people plan to be at the public place and the security level chosen by the government.

We consider a strategic population where each person chooses whether or not to expose themselves to terror risk. For instance, people choose to stay at home or go to a concert, live in the suburbs or the city, and travel by car or use public transport. In all of these cases, people choose between a relatively safer option with very low terrorism threat and one with high terror threat. By attending

[5] We focus on defensive counter-terrorism security (e.g., Bandyopadhyay and Sandler (2011), Brandt and Sandler (2011)) rather than proactive counter-terrorism security. Hence, whenever we use the term "security," it should be understood as "defensive security."

[6] In our paper, the term "public place" is not restricted only to physical places but also includes any event or activity that is open to the public and potentially attracts the crowds (e.g., concert, public transportation, market place, tourist attraction).

[7] Some authors, for example, Abrahms (2008), posit that terrorist need not be rational agents.

the public place, each person takes the risk of not getting any value if the terror attack is successful or getting a high value in case it is not. By not attending the public place, population members get their respective reservation values. The choice of the population members depends on the government's level of counter-terrorism measures and terrorist's effort.

In the simultaneous game (with fixed security level) between terrorist and population, we find that the mass of the population that exposes itself to terror risk is always increasing in the security level while the probability of a successful attack is always decreasing in the security level. However, when it comes to the terror effort, the results are more complex as the terror effort is either decreasing (expected result) or increasing (unexpected result) in the security level. This is because, counter-terrorism security acts as both deterrent and encouragement for terrorist. On the one hand, more security makes it less likely that the attack will be successful. On the other hand, more security motivates more people to attend the public place; this increase in attendance makes that very same public place a more attractive target for a terror attack.

Next, we turn our attention to the main problem of this paper: effectiveness of the government-provided security in the context of saving lives. Terrorist attacks have many negative consequences: destruction of infrastructure, decrease in foreign direct investment, disruption of financial markets, etc. While infrastructure can be rebuilt, foreign investors can return, and financial markets can bounce back, there is one target of terror attacks that can not be recovered: human life. For that reason terrorism is currently such an important issue and that is why we focus on lost lives.

The relevant measure to consider is the expected number of casualties due to terror attack. Now, we can more precisely posit our main question:

Suppose that the counter-terrorism security is costless. If the goal is to decrease the expected number of casualties, is more security always better?

Under standard and general assumptions, we find that the relationship between the security level and expected number of casualties can be summarized in the three cases that we depict in Figure 1. Case 1 is what we desire and instinctively predict: the expected number of casualties always decreases in security. In the second case, we have the very opposite since the expected number of casualties always increases. It is Case 3 – where the expected number of casualties increases if security is too small and decreases when security is large enough (inverted U-shaped relationship) – that we find to be the most interesting and important.

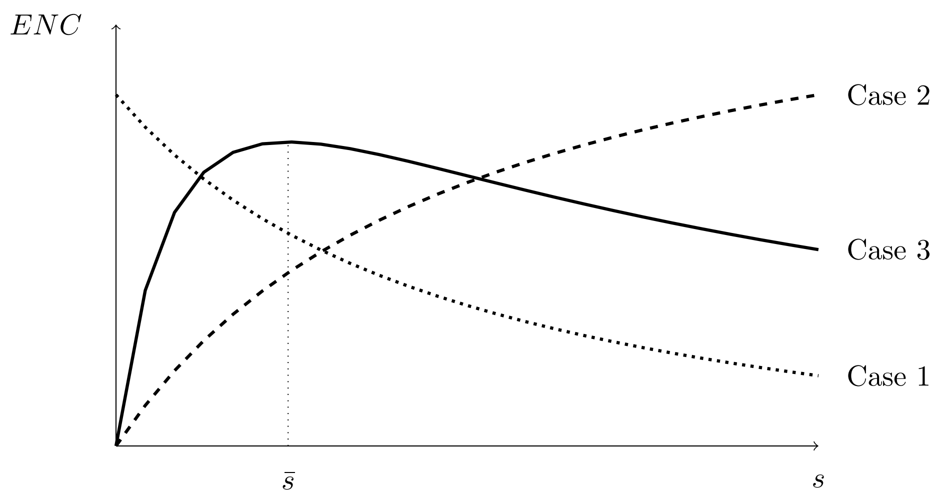

Figure 1: Counter-terrorism security (s) and expected number of casualties (ENC).

Case 3 supports the view that it is better to over-spend on counter-terrorism measures rather than to act too prudently. When the government spends too little (i.e., less than a threshold value $\bar{s}$), then increasing security will actually bring devastating results. However, there is no such risk with increasing the expected number of casualties by expanding the counter-terrorism budget when the expenditures are already above the threshold level. Case 3 is also important for the government that operates under sever constraints; that is, when the maximum feasible amount of security is at most $\bar{s}$. With such a constraint, Case 3 is the same as Case 2 where the optimal quantity of security is zero.

In short, the answer to our question – whether more counter-terrorism security is always better – is that more is not always better and, actually, more can make things worse.

Related literature. The impact of terrorism on behavior has also been studied in the literature; Sandler and Enders (2008) provide an excellent review of economic consequences of terrorism. Our focus on "public places" is motivated by Brandt and Sandler (2011) who show that terrorists are now concentrating on attacking public places. Enders and Sandler (1991), Enders et al. (1992) and Drakos and Kutan (2003) find evidence of tourists not favoring destinations with higher risk of terrorism. Elias et al. (2013) show that commuters in Israel choose private means of transport over public transport as their risk perception of terror activities increase. Kalist (2010) documents that in the US, as the terror-alert levels initially went up after the September 2001 attacks, there was a dip in attendance for Major League Baseball events. Abadie and Gardeazabal (2008) show how investors strategically move away from countries and regions with higher terrorist risk.

There has been a lot of game theoretic work on terrorism in the last couple decades. Excellent reviews of this literature can be found in Sandler and Arce (2003), Bueno De Mesquita (2008), and Sandler and Siqueira (2009). Many game theoretic models of terrorism consider the case where the government acts first and chooses its security level which negatively impacts the effectiveness of terrorist activity followed by the terrorists choosing their action. Papers where the government acts first and allocates resources towards security include Bueno De Mesquita (2005), Bueno De Mesquita (2007), Bandyopadhyay et al. (2011), and Carter (2015).

A strand of literature considers the case of a defender protecting multiple locations against an attacker. Powell (2007a) and Powell (2007b) both consider the case of a defender choosing how to divide a fixed security resource across sites and attacker choosing how to divide the probability of attacking each of those sites. Bier et al. (2007) also consider optimal security provision across two sites. They model the probability of a successful attack as a function of government security for that site and allow the terrorist to choose which site to attack. Zhuang and Bier (2007) allow the probability of successful attack across multiple cites to vary with security level and terrorist effort as in this paper. Wang and Bier (2011) model a situation with multiple targets with incomplete information where the defender is unaware of which attacker's private preferences over sites. An important result of the literature till now is that government allocates security such that the marginal loss from the two sites is equated. Many other papers consider government security provision in the presence of a strategic terrorist including notable work by Rosendorff and Sandler (2004), Siqueira and Sandler (2006), Sandler and Siqueira (2006), Farrow (2007), Bueno

De Mesquita (2007), and Bandyopadhyay and Sandler (2011).

Keohane and Zeckhauser (2003) consider a game theoretic model with a strategic population which chooses to remove itself from the site of intended terror attack. Given a fixed disutility of harm from terror attack and a cost of switching to a safe haven, they show that if probability of a successful attack is increasing in the number of people at the location then the number of people at the site of the terror attack will be such that the harm from terror is just equal to the cost of the other location. In this homogenous setting, government measures to reduce the probability of harm or the disutility from harm are both ineffective to change the total welfare. Our model offers very different insight by including government, terrorist and population as strategic players in a multi-stage game and by further allowing for heterogeneity in the population. An additional point of departure of this paper is in considering the government which tries to maximize the sum of utilities for entire population instead of just trying to minimize the impact of the terrorist. Another paper looking at a strategic population is Lakdawalla and Zanjani (2005) where residents choose between safe and risky options. They look at the case where the choice of safe options by individuals creates a public bad which can be avoided by the government giving incentives for insurance.

The next section describes the model. Section 2 looks at the second-stage game between the terrorist and the population. Section 4 analyzes the effectiveness of counter-terrorism in terms of saving lives and welfare. Section 5 concludes. All proofs are collected in the Appendix.

2 Model

The three players of the game are the government, terrorist and population. The game consists of two rounds such that:

- In the first round government chooses counter-terrorism security level.
- The second round consists of a simultaneous game involving the terrorist and population members. Members of the population choose whether to attend the public place or not while

the terrorist chooses the effort with which to attack that public place.

The game begins with the government choosing the level of security denoted by s. Next, the terrorist and members of the population act. We let t denote the level of effort chosen by the terrorist. Members of the population individually decide whether or not to attend the public place. We let m denote the total mass of population that chooses to attend the public place.

A fundamental variable in our model is the probability of a successful attack denoted by $\lambda(s,t)$. We assume that the terrorist and government both have control over this variable. The government can decrease the probability of a successful attack by increasing its security effort while the terrorist can increase that probability by increasing their terror effort. More formally, we make the following assumptions about the $\lambda(s,t)$. Note, however, that we make no assumptions about the functional form of $\lambda(s,t)$.

Assumption 1.

$\lambda(s,t)$ is continuous in both s and t and twice continuously differentiable such that $\lambda_t > 0$, $\lambda_s < 0$, $\lambda_{tt} \leq 0$, and $\lambda_{ts} < 0$.[8]

The assumptions on the probability of the successful attack ensure that the probability is increasing at a decreasing rate in the terrorist effort but decreasing in the government-provided security. Additionally, the marginal gain from terrorist effort, is decreasing in s. Hence government security dampens the overall probability of a successful attack and also the marginal impact from terrorist effort.

The terrorist seeks to maximize the expected impact of terror less the cost of terror. The members of population choose whether the utility of attending the place given the probability of a successful attack is more than that of staying away. We next discuss the terrorist's and the population's problems in detail.

Terrorist. We model a rational terrorist who maximizes a utility function which in the standard form: benefit from the terror act less the cost of execution of that act. We assume that both

[8]The notation used in this paper conforms to standard notation where given a function $f(x,y)$ of variables x and y, we write $f_x = \frac{\partial f}{\partial x}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, and $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$.

benefit and cost are measured in psychological units. A plausible assumption regarding the benefit from the terror act is that the terrorist's utility is rising in the expected impact of the act. We simplify this argument and assume that the terrorist benefit is exactly the expected impact of the act. We do not assume any particular functional form of the cost of the terror act except for what is typical for the literature: cost is increasing and convex in effort.

Assumption 2.

$c(t)$ is continuous and twice continuously differentiable in t such that $c' > 0$ and $c'' > 0$.

The expected impact of terror depends on the probability of a successful attack and the size of the population that attends the public place. Let m denote the mass of population attending the public place and $c(t)$ denote the cost of effort. The terrorist maximizes expected utility U_T.

$$U_T(s, t, m) = m \cdot \lambda(s, t) - c(t) \tag{1}$$

The terrorist gets utility zero if the attack is unsuccessful while if the attack is successful the terrorist's utility is exactly m which is the mass of population that was attacked. This utility captures the fact that the terrorist not only wants to launch a successful attack but also inflict maximum possible damage. We assume that the terrorist's reservation utility is zero. This implies that the terror attacks whenever $U_T \geq 0$.

The following standard technical assumption regarding the terrorist's utility function ensures that the terrorist's optimization program has an interior solution.

Assumption 3.

$\lim_{t\to 0} [m\lambda_t - c'] > 0$ *and* $\lim_{t\to\infty} [m\lambda_t - c'] < 0$.

Population. We model a strategic population of total mass one. Each member of the population has two choices — of either attending the public place and exposing themselves to terror risk or of staying away from the public place. Each member evaluates the relative utility from going to the public place versus staying away and chooses the option that gives them the higher utility.

To evaluate the utility from going to the public place, each member must consider two cases. One where the terror attack is successful and the other where it is not successful. We assume that everyone gets utility one from attending the public place in case the attack fails while if a person attends a public place and the attack is successful, then the utility is zero. Since the members of the population are maximizing expected utility, they evaluate the expected benefit from going to the public place as $1 \cdot (1 - \lambda(s,t)) + 0 \cdot (\lambda(s,t))$. Each member thus has a rational expectation of the probability of a successful attack $\lambda(s,t)$.[9]

Instead of attending the public place, an individual can stay at home. Here, we assume that the population is heterogenous in terms of utility derived from staying at home. That is, the value of staying at home is not the same for everyone. Let $\alpha \in [0,1]$ denote the utility from staying at home. We make no specific assumption about the variable α other than it is distributed over $[0,1]$ with cumulative distribution function F. Let F_α denote the first-order derivative of F, $F_\alpha := \frac{dF}{d\alpha}$. We assume that F is continuous.

Assumption 4.

F is continuous.

We can think of these utility values — zero, α, and one — as being relative values. That is, we normalize the utility function in such a way that its range is between zero and one. Each individual faces three scenarios: go to the public place and not face a terror attach (highest utility value of 1), go to the public place and face a terror attack (lowest utility value of 0), or stay away from the public place (intermediate utility $\alpha \in [0,1]$). Each member of the population chooses to go to the public place only if the value from staying away is low enough given the probability of a successful terror attack.

Government. We analyze the impact of increasing counter-terrorism security on the **Expected Number of Casualties**, $m \cdot \lambda$. We also look at how security alters social welfare, U_{TW}, where social welfare is the sum of two elements: the aggregate utility of those who attend the public place, $U_1(s,t,m) = m(1 - \lambda(s,t))$, and the aggregate utility of those who stay at home, $U_2(s,t,m) =$

[9]We can think of members of the population forming this expectation via information from news media.

$\int_{\bar{\alpha}}^{1} \alpha dF$.

We solve the game by backward induction. In the next section, we solve for the second-stage of the game where government security level is fixed and the members of population and terrorist choose their strategies simultaneously. We characterize the Nash equilibrium for the second stage as a function of the level of government security. In Section 4, we then consider the problem of government choosing the security level.

3 Population and Terrorist

We begin by analysing the behaviour of the terrorist and the population given a fixed level of security s. That is, we find the equilibrium in the second-stage simultaneous-move game between population and terrorist. In this section, the government-provided security level is presumed to be fixed.

Every person chooses between staying at home and attending the public place. Staying at home generates certain utility α while attending the public place generates a lottery that yields utility zero with probability λ and utility one with probability $1-\lambda$. Hence, the expected utility from attending the public place is $1-\lambda$.[10]

Given s and t, let $\bar{\alpha}(s,t)$ be the highest utility from staying at home such that everyone with utility from home less than $\bar{\alpha}$ attends the public place. Since every person maximizes expected utility, $\bar{\alpha}(s,t)$ is defined in the following way.

$$\bar{\alpha}(s,t) \quad = \quad 1-\lambda(s,t) \tag{2}$$

Let $\hat{m}(s,t)$ denote the mass of the population that attends the public place given that the terrorist's

[10] We assume that if a person is indifferent between staying at home (utility α) and attending the public place (expected utility $1-\lambda$), then that person attends the public place.

effort is t.

$$\hat{m}(s,t) = F(\bar{\alpha}(s,t)) = F(1-\lambda(s,t)) \tag{3}$$

Next, let $\hat{t}(s,m)$ denote terrorist's best-response function given the mass of the population attending the public place, m. There exists a unique interior solution of the terrorist's optimization program (Equation (1)). This solution, $\hat{t}(s,m)$, satisfies the following first-order condition.

$$\frac{\partial U_T}{\partial t} = m\lambda_t - c' = 0 \tag{4}$$

We turn attention to the second-stage equilibrium values, $\tilde{m}(s)$ and $\tilde{t}(s)$. Both $\tilde{m}(s)$ and $\tilde{t}(s)$ are functions of counter-terrorism security s since that security is chosen at the fist stage (hence, s is fixed at the second stage). We derive $\tilde{m}(s)$ and $\tilde{t}(s)$ from the following system of two equations.

$$\tilde{m}(s) = \hat{m}(s,\tilde{t}(s)) \tag{5}$$

$$\tilde{t}(s) = \hat{t}(s,\tilde{m}(s)) \tag{6}$$

Our first proposition analyses the behavior of $\tilde{m}(s)$. The result shows that government security measures encourage more people to attend the public place and increases in government spending do not lead to any reduction in the mass of people attending the public place; that is, $\frac{d\tilde{m}(s)}{ds} \geq 0$ for each s. In our model, a member of the population who chooses to go to the public place at a lower government security level continues to choose to go to the public place when the government security level increases.

Proposition A.

The second-stage equilibrium mass of the population that attends the public place $\tilde{m}(s)$ is weakly increasing in security s for each s.

Proposition A shows that the equilibrium behavior of the population is such that the mass of population attending the public lace weakly increases with security level. The formal proof of this proposition is in the Appendix where, first, we show that the best-response function of the terrorist, $\hat{t}(s, m)$, and the function capturing the mass of population that attends the public place for a given level of terrorist effort and government security, $\hat{m}(s, t)$, both behave in the expected manner. That is, $\hat{t}(s, m)$ is decreasing in security s but increasing in the size of population attending the public place m while $\hat{m}(s, t)$ is increasing in security s but decreasing in terrorist effort t.

Proposition A presents a result about the equilibrium behavior of the population which takes into account the equilibrium behavior of the terrorist. The intuition for this result lies in two facts based on the best-response functions. First, we know that keeping terrorist activity constant, the mass of population attending the public place is increasing with security. Second, increased security leads to the terrorist reducing effort for each fixed size of population and this reduction in terrorist effort leads to a greater mass of population attending the public place.

The next proposition analyzes the behavior of $\tilde{t}(s)$. This result shows that the relationship between the equilibrium terrorist action and government security is not as clear cut as that of the population members. As government spending increases, the effectiveness of terrorist effort decreases and we can think of this as causing the terrorist to expend lower effort. But then at the same time, the mass of population at the public place increases as government spending increases (Proposition A) and this motivates the terrorist to expend more effort. Hence, the change in the second-stage equilibrium level of terrorist effort may increase or decrease as government security increases.

Proposition B.

1. *If* $F(1-\lambda) \cdot \lambda_{ts} - F_\alpha(1-\lambda) \cdot \lambda_s \cdot \lambda_t \geq 0$ *for each pair* (s, t)*, then the second-stage equilibrium terrorist effort* $\tilde{t}(s)$ *is weakly increasing in security* s *for each* s*.*

2. *If* $F(1-\lambda) \cdot \lambda_{ts} - F_\alpha(1-\lambda) \cdot \lambda_s \cdot \lambda_t \leq 0$ *for each pair* (s, t)*, then the second-stage equilibrium terrorist effort* $\tilde{t}(s)$ *is weakly decreasing in security* s *for each* s*.*

Proposition B shows that the equilibrium behavior of the terrorist is not always decreasing (or increasing) with security. As with Proposition A, the intuition behind this result lies in two facts based on the best-response functions. First, we know that keeping the mass of population attending the public place constant, increased security leads to reduced terror effort. Second, increased security leads to a greater mass of population attending the public place and the terrorist effort is increasing in the mass of population attending the public place for a fixed level of security. Hence, whether terrorist effort increases or decreases with government security level depends on which effect is greater.

We present two numerical examples whose only purpose is to indicate that the inequality in Proposition B might and might not hold.

Example 3.1.

Case 1 in Proposition B: $\frac{d\tilde{t}(s)}{ds} \geq 0$ *for each* s. *Assume that* $\lambda(s,t) = \frac{1}{2(1+s)}\left(1 - e^{-t}\right) + \frac{1}{2}$ *and* α *is uniformly distributed. Note that* $\lambda(s,t) \in [0.5, 1]$ *for each pair* (s,t). *In order to determine the sign of* $\frac{d\tilde{t}(s)}{ds}$, *we compute* $F(1-\lambda) \cdot \lambda_{ts} - F_\alpha(1-\lambda) \cdot \lambda_s \cdot \lambda_t = \frac{1}{2}(1+s)^{-2}e^{-t}(2\lambda - 1)$. *Since* $\lambda \geq \frac{1}{2}$, *it is true that* $F(1-\lambda) \cdot \lambda_{ts} - F_\alpha(1-\lambda) \cdot \lambda_s \cdot \lambda_t \geq 0$. *Consequently,* $\frac{d\tilde{t}(s)}{ds} \geq 0$ *for each* s.

Example 3.2.

Case 2 in Proposition B: $\frac{d\tilde{t}(s)}{ds} < 0$ *for each* s. *Assume that* $\lambda(s,t) = \frac{1}{2(1+s)}\left(1 - e^{-t}\right)$ *and, again,* α *is uniformly distributed. Note that* $\lambda(s,t) \in [0, 0.5]$ *for each pair* (s,t). *In order to determine the sign of* $\frac{d\tilde{t}(s)}{ds}$, *we compute* $F(1-\lambda) \cdot \lambda_{ts} - F_\alpha(1-\lambda) \cdot \lambda_s \cdot \lambda_t = \frac{1}{2}(1+s)^{-2}e^{-t}(2\lambda - 1)$. *Since* $\lambda \leq \frac{1}{2}$, *it is true that* $F(1-\lambda) \cdot \lambda_{ts} - F_\alpha(1-\lambda) \cdot \lambda_s \cdot \lambda_t \leq 0$. *Consequently,* $\frac{d\tilde{t}(s)}{ds} \leq 0$ *for each* s.

Finally, we analyze the behavior of the probability of successful attack computed for the optimal level of terror effort, $\tilde{\lambda}(s) := \lambda(s, \tilde{t}(s))$. Proposition C shows that, as expected, the probability of a successful attack given the equilibrium level of terrorist effort is decreasing in the government security level.

Proposition C.

The probability of a successful attack $\tilde{\lambda}(s)$ *is strictly decreasing in security* s *for each* s.

Proposition C is important because it shows that government security works in the right direction at equilibrium by reducing the equilibrium probability of a successful attack. We know that the probability of a successful attack responds directly to changes in security level and also indirectly via the terrorist level. From our assumptions, we know that the direct impact of security level is to reduce the probability of successful attack. From Proposition B, we know that, at equilibrium, the impact of security might be to increase or decrease the terror effort which in turn is positively correlated with the probability of successful attack. Proposition C shows that even if terrorist effort increases in response to increased government security, the overall impact is that the probability of a successful attack goes down.

4 Effectiveness of Counter-Terrorism Security

We arrive at the core analysis of our paper: the effectiveness of counter-terrorism security. Our main focus is the situation where the government chooses counter-terrorism security to minimize the expected impact of terror where that impact is measured by the expected number of casualties; i.e., the mass of population at the public place times the probability of a successful attack, $\tilde{m}(s) \cdot \tilde{\lambda}(s)$.

The government-provided security always decreases the probability of successful terror attack (Proposition C) but, at the same time, it also increases the size of the population attending the public place (Proposition A) and either increases or decreases the terror effort (Proposition B). Consequently, it need not be the case that more security is always better.

Proposition D is our first results that analyzes the effectiveness of the counter-terrorism security. This result shows that the relationship between security and expected number of casualties is not as straightforward as we instinctively might think (i.e., more security always decreases the expected number of casualties) as more is not always better and can actually be worse.

Proposition D.

1. *If* $F(1-\lambda) - F_{\alpha}(1-\lambda) \cdot \lambda \leq 0$ *for each pair* (s, t)*, then the expected number of casualties* $\tilde{m}(s) \cdot \tilde{\lambda}(s)$ *is weakly increasing for each* s*.*

2. *If* $F(1-\lambda) - F_{\alpha}(1-\lambda) \cdot \lambda \geq 0$ *for each pair* (s, t)*, then the expected number of casualties* $\tilde{m}(s) \cdot \tilde{\lambda}(s)$ *is weakly decreasing for each* s*.*

The intuition for Proposition D comes from the fact that increased security leads to a reduction in the equilibrium probability of a successful attack (Proposition C) and leads to an increase in the equilibrium mass of population attending the public place (Proposition A). These two effects of security work in opposite directions and it is not clear whether increased security leads to a greater/lesser expected number of casualties. Given that increased security leads to a reduction of the probability of a successful attack, it encourages more individuals to attend the public place. Hence, it might be the case that increases security encourages too many individuals to attend the public place which negates the effect of the lower probability of a successful attack and leads to a higher expected damage from terrorism.

So far, in Proposition D, we show that it is possible that the expected number of casualties increases when the counter-terrorism security increases. However, our result lacks some regularity. After all, Proposition D does not reject the possibility that the expected number of casualties initially increases, then decreases, then increases again, and so on. In other words, it is possible that the range of values of security s looks like a patchwork with each segment having different behavior of the expected number of casualties compared to the adjacent segments.

Our goal is to achieve some form of regularity. That is, we want the range of values of s to be divided into at most two segments: one on which the expected number of casualties is increasing and one on which the expected number of casualties is decreasing. To that end, we enrich our model and add Assumption 4.

Assumption 5.

F is log-concave.

Log-concavity is a standard and commonly made assumption in the literature (see Bagnoli and Bergstrom (2005)).

From Proposition D, we know that the expected number of casualties is weakly increasing if and only if the following holds.

$$\frac{1}{\tilde{\lambda}} \leq \frac{F_{\alpha}(1-\tilde{\lambda})}{F(1-\tilde{\lambda})} \tag{7}$$

Our objective is to analyze the inequality in (7). First, from Proposition C, we know that $\tilde{\lambda}$ is a strictly decreasing function. This implies that the left-hand side of inequality in (7) is a strictly increasing function.

To analyze the right-hand side of inequality (7) note that, since $1-\tilde{\lambda}$ is an increasing function, it is true that $F(1-\tilde{\lambda})$ is also an increasing function in s. Now, to get sharper results, we add Assumption 4; namely, F is log-concave. Log-concavity implies that $\frac{F_\alpha(1-\tilde{\lambda})}{F(1-\tilde{\lambda})}$ is a decreasing function in s.

To summarize, on the left-hand side of inequality in (7), there is a strictly increasing function. On the right-hand side, the function is decreasing. Hence, these functions can cross at most (if at all) only once. If they do not cross, then the inequality in (7) either holds for each s (i.e., the expected number of casualties always increases) or does not hold for any s (i.e., the expected number of casualties always decreases).

But suppose that they do cross at $\bar{s}$. Then $\frac{1}{\tilde{\lambda}} \leq \frac{F_\alpha(1-\tilde{\lambda})}{F(1-\tilde{\lambda})}$ for all $s<\bar{s}$ and $\frac{1}{\tilde{\lambda}} \leq \frac{F_\alpha(1-\tilde{\lambda})}{F(1-\tilde{\lambda})}$ for all $s>\bar{s}$. That is, the expected number of casualties initially increases to start decreasing for $s>\bar{s}$.

Our analysis is summarized in Proposition E which shows that there are just three cases describing the relationship between counter-terrorism security and expected number of casualties.

Proposition E.

If we add Assumption 4, then only one of the following is true.

Case 1. The expected number of casualties $\tilde{m}(s)\cdot\tilde{\lambda}(s)$ is weakly decreasing in security s for each s.

Case 2. The expected number of casualties $\tilde{m}(s)\cdot\tilde{\lambda}(s)$ is weakly increasing in security s for each s.

Case 3. There exists security level $\bar{s}$ such that the expected number of casualties $\tilde{m}(s)\cdot\tilde{\lambda}(s)$ is weakly increasing in security s for each $s<\bar{s}$ and weakly decreasing in security s for each $s>\bar{s}$.

For each case in Proposition E, we provide a numerical example. Each example is characterized by the following.

1. $\lambda(s,t) = \frac{1}{1+s}t + x$ where $x = 0$ for Case 1, $x = 0.75$ for Case 2, and $x = 0.25$ for Case 3.

2. Cost function $c(t) = \frac{1}{2}t^2$.

3. α is uniformly distributed.

Note that $\tilde{t}(s) = \frac{(1-x)(1+s)}{(1+s)^2+1}$, $\tilde{m}(s) = \frac{(1-x)(1+s)^2}{(1+s)^2+1}$, and $\tilde{\lambda}(s) = \frac{x(1+s)^2+1}{(1+s)^2+1}$. It is easy to check that $\tilde{m}(s) \cdot \tilde{\lambda}(s)$ is strictly decreasing if $x = 0$ and strictly increasing if $x = 0.75$. If $x = 0.25$, then $\tilde{m}(s) \cdot \tilde{\lambda}(s)$ is strictly increasing up to $\bar{s} = 0.4142$ and strictly decreasing for $s > \bar{s}$.

The first case captures the expected and desired impact of security: more is always better. The second case is the extreme opposite of Case 1. It is an unexpected result as Case 2 indicates that even under costless security the optimal security level should be zero because more security will always result with more casualties. In Case 2, more is always worse.

Case 3 is between the two extremes depicted in Case 1 and Case 3. We find this case especially interesting and important. For small values of government security, the expected impact of terror is rising, rather than decreasing, in government security. However, this expectation is decreasing for large enough values of government security. That is, the counter-productive security can happen only when that security is too small.

Our last result analyzes the impact of security on total welfare. Total welfare sums over the welfare for all members of the population and the next proposition shows that increasing government security increases the welfare.

Proposition F.

Total welfare weakly increases in security s for each s.

Proposition F can be intuitively explained in the following way. When security increases, then our population can be divided into three groups. First, there are people who were attending the public place before the increase of security. They continue attending the public place and their total welfare increases as the probability of successful attack decreases (Proposition C). Second, due to Proposition A, we know that there is a group of people who switch from staying away to attending the public place. For this group, the total welfare also increases; after all, the opt for attending the public place because this is a more attractive alternative compared to staying away. Finally, the third group: people who continue staying away after security increased. For them, nothing changes in terms of utility. Overall, we note that increasing security either increases utility or keeps utility at the same level. Consequently, total welfare is not strictly decreasing.

4.1 Policy Implications

Results in Proposition E, especially Case 3, are very important from the policy perspective. Contrary to what intuition might suggest, Proposition E tells that the expected number of casualties need not always be decreasing in security; an increase is also plausible.

If Case 3 is the one that captures reality, then knowing the threshold security level, $\bar{s}$, implies the certainty that increasing security beyond that threshold value will always, as desired, decrease the expected number of casualties. In short, Case 3 says that it is better to over-spend on counter-terrorism than under-spend.

Overspending has been found in the game-theoretic literature looking at defensive security in Lapan and Sandler (1988) and Sandler and Siqueira (2006). We show that overspending might in fact be optimal even if there is no other country to which the terror attack might be diverted. Our results also highlight that if the government is constrained as to the amount of security it can provide, such that its funds allow provision of $s < \bar{s}$ then it might be optimal to provide no security at all. Some security can in fact be more harmful than no security.

We feel that the results in our paper can help rationalize the varying security levels chosen by countries facing a similar terror risk. We show that a government with heavy resource constraints should always choose a very low level of security whereas a government with no significant constraints should choose a security level high enough to negate any incentives to increase the terror effort steaming from the greater mass of population attending the public place.

5 Conclusions

Terrorism is one of the biggest concerns of the modern world and has a significant impact on our daily behaviors. In response to the threat of terrorism, governments spend resources on providing counter-terrorism security. In this paper, we analyze the effectiveness of this security as measured by the expected number of casualties due to terror attack.

We make a theoretical contribution to the literature by the addition of a strategic population. We find that increasing security need not lead to a decrease of the expected number of casualties. In fact, it is likely that this number would decrease; that is, more security causes more harm. It also is possible that this unexpected outcome happens only when security is too small.

Our results have important policy implications. In the case of the government which wants to reduce the expected impact of terror but operates under constraints (i.e., the limit of how much security can be provided is low), we show that less security might in fact be better if the government is unable to provide infinite security. However, even if there are no severe constraints, the government must be cautious not to be too prudent as over-spending on counter-terrorism security could be significantly better than under-spending.

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A Appendix: Proofs

Preliminary Results. We start with results that will help us prove the main propositions.

Recall that, for a fixed s, $\hat{m}(s,t)$ denotes population's best-response function to terror effort t. Equation (3) depicts $\hat{m}$ and Lemma 1 describes the behavior of $\hat{m}$.

Lemma 1.

a) For each s, $\frac{\partial \hat{m}}{\partial t} \leq 0$.

b) For each t, $\frac{\partial \hat{m}}{\partial s} \geq 0$.

Proof of Lemma 1.

(a) Observe the following.

$$\frac{\partial \hat{m}}{\partial t} = -F_\alpha(1-\lambda) \cdot \lambda_t \tag{8}$$

Since $F_\alpha \geq 0$ and $\lambda_t > 0$ (Assumption 1), we conclude that $\frac{\partial \hat{m}}{\partial t} \leq 0$.

(b) Observe the following.

$$\frac{\partial \hat{m}}{\partial s} = -F_\alpha(1-\lambda) \cdot \lambda_s \tag{9}$$

Since $F_\alpha \geq 0$ and $\lambda_s < 0$ (Assumption 1), we conclude that $\frac{\partial \hat{m}}{\partial s} \geq 0$. ■

Recall that, for a fixed s, $\hat{t}(s,m)$ denotes terrorist's best-response function to population's mass m. Equation (4) depicts $\hat{t}$ and Lemma 2 describes the behavior of $\hat{t}$.

Lemma 2.

a) For each s, $\frac{\partial \hat{t}}{\partial m} > 0$.

b) For each m, $\frac{\partial \hat{t}}{\partial s} < 0$.

Proof of Lemma 2.

(a) Observe the following.

$$\frac{\partial \hat{t}}{\partial m} = \frac{-\lambda_t}{m \cdot \lambda_{tt} - c''} \tag{10}$$

Since $\lambda_t > 0$, $\lambda_{tt} \leq 0$, and $c'' > 0$ (Assumption 1), we conclude that $\frac{\partial \hat{t}(s,m)}{\partial m} > 0$.

(b) Observe the following.

$$\frac{\partial \hat{t}}{\partial s} = \frac{-m \cdot \lambda_{ts}}{m \cdot \lambda_{tt} - c''} \tag{11}$$

Since $\lambda_{ts} < 0$, $\lambda_{tt} \leq 0$, and $c'' > 0$ (Assumption 1), we conclude that $\frac{\partial \hat{t}(s,t)}{\partial s} < 0$. ■

Recall that the system of two equations (5) and (6) depicts the second-stage equilibrium values $\tilde{m}(s)$ and $\tilde{t}(s)$ which we analyze in Propositions A and B, respectively. In order to prove these propositions, we need the first-order derivatives of $\tilde{m}$ and $\tilde{t}$ with respect to s.

$$\frac{d\tilde{m}}{ds} = \frac{\partial \hat{m}}{\partial s} + \frac{\partial \hat{m}}{\partial t} \cdot \frac{d\tilde{t}}{ds} \tag{12}$$

$$\frac{d\tilde{t}}{ds} = \frac{\partial \hat{t}}{\partial s} + \frac{\partial \hat{t}}{\partial m} \cdot \frac{d\tilde{m}}{ds} \tag{13}$$

We re-write the above in the following way.

$$\left[1 - \frac{\partial \hat{m}}{\partial t}\frac{\partial \hat{t}}{\partial m}\right]\frac{d\tilde{m}}{ds} = \frac{\partial \hat{m}}{\partial s} + \frac{\partial \hat{m}}{\partial t} \cdot \frac{\partial \hat{t}}{\partial s} \tag{14}$$

$$\left[1 - \frac{\partial \hat{t}}{\partial m}\frac{\partial \hat{m}}{\partial t}\right]\frac{d\tilde{t}}{ds} = \frac{\partial \hat{t}}{\partial s} + \frac{\partial \hat{t}}{\partial m} \cdot \frac{\partial \hat{m}}{\partial s} \tag{15}$$

To help with the proofs, we re-write $\frac{\partial \hat{t}}{\partial m}$ and $\frac{\partial \hat{t}}{\partial s}$ in the following way.

$$\frac{\partial \hat{t}}{\partial m} = \frac{\lambda_t^2}{c'' \cdot \lambda_t - c' \cdot \lambda_{tt}} \tag{16}$$

$$\frac{\partial \hat{t}}{\partial s} = \frac{c' \cdot \lambda_{ts}}{c'' \cdot \lambda_t - c' \cdot \lambda_{tt}} \tag{17}$$

In order to simplify the notation, we write F and F_λ instead of $F(1-\lambda)$ and $F_\alpha(1-\lambda)$, respectively.

Proof of Proposition A. We want to prove that $\frac{d\tilde{m}(s)}{ds} \geq 0$ for each s. We analyze $\frac{d\tilde{m}}{ds}$ by looking at equation (14).

(i) Left-hand side of (14). Since $\frac{\partial \hat{m}}{\partial t} \leq 0$ (Lemma 1) and $\frac{\partial \hat{t}}{\partial m} > 0$ (Lemma 2), we conclude that $\frac{\partial \hat{m}}{\partial t}\frac{\partial \hat{t}}{\partial m} \leq 0$. Hence, $\left[1 - \frac{\partial \hat{m}}{\partial t}\frac{\partial \hat{t}}{\partial m}\right] > 0$. This implies that the sign of $\frac{d\tilde{m}}{ds}$ is the same as sign of the right-hand side of equation (14).

(ii) Right-hand side of (14). Since $\frac{\partial \hat{m}}{\partial s} \geq 0$ (Lemma 1), $\frac{\partial \hat{m}}{\partial t} \leq 0$ (Lemma 1), and $\frac{\partial \hat{t}}{\partial s} < 0$ (Lemma 2), we conclude that the right-hand side of equation (14) is at least zero. From (i) and (ii), we conclude that $\frac{d\tilde{m}}{ds} \geq 0$. ■

Proof of Proposition B. We only prove (1) as the proof of (2) follows the same reasoning. We want to prove that $\frac{d\tilde{t}(s)}{ds} \geq 0$ for each s if $F \cdot \lambda_{ts} - F_\alpha \cdot \lambda_s \cdot \lambda_t \geq 0$ for each pair (s,t). We derive $\frac{d\tilde{t}}{ds}$ from equation (15).

(i) Left-hand side of (15). Since $\frac{\partial \hat{t}}{\partial m} > 0$ (Lemma 2) and $\frac{\partial \hat{m}}{\partial t} \leq 0$ (Lemma 1), we conclude that $\frac{\partial \hat{t}}{\partial m}\frac{\partial \hat{m}}{\partial t} \leq 0$. Hence, $\left[1 - \frac{\partial \hat{t}}{\partial m}\frac{\partial \hat{m}}{\partial t}\right] > 0$. This implies that the sign of $\frac{d\tilde{t}}{ds}$ is the same as the sign of the right-hand side of equation (15).

(ii) Right-hand side of (15). In order to determine the sign of the right-hand side of equation (15), we use equations (9), (16), and (17) and write $\frac{\partial \hat{t}}{\partial s} + \frac{\partial \hat{t}}{\partial m}\frac{\partial \hat{m}}{\partial s} = \frac{c' \cdot \lambda_{ts} - F_\alpha \cdot \lambda_s \cdot \lambda_t^2}{c'' \cdot \lambda_t - c' \cdot \lambda_{tt}}$. Given Assumptions 1 and 2, we conclude that the sign of $\frac{d\tilde{t}}{ds}$ is the same as the sign of $c' \cdot \lambda_{ts} - F_\alpha \cdot \lambda_s \cdot \lambda_t^2$. From the terrorist's first-order condition, we know that $c' = \tilde{m} \cdot \lambda_t = F \cdot \lambda_t$. Hence, $c' \cdot \lambda_{ts} - F_\alpha \cdot \lambda_s \cdot \lambda_t^2 = \lambda_t \cdot [F \cdot \lambda_{ts} - F_\alpha \cdot \lambda_s \cdot \lambda_t]$. Since $\lambda_t > 0$ (Assumption 1), we conclude that if $F \cdot \lambda_{ts} - F_\alpha \cdot \lambda_s \cdot \lambda_t \geq 0$ for each pair (s,t), then $\frac{d\tilde{t}}{ds} \geq 0$ for each s. ■

Proof of Proposition C. We want to prove that $\frac{d\tilde{\lambda}(s)}{ds} < 0$ for each s. Observe that $\frac{d\tilde{\lambda}}{ds} = \lambda_s + \lambda_t \frac{d\tilde{t}}{ds}$. We use equation (13), and write $\frac{d\tilde{\lambda}}{ds} = \lambda_s + \lambda_t \left[\frac{\partial \hat{t}}{\partial s} + \frac{\partial \hat{t}}{\partial m} \cdot \frac{d\tilde{m}}{ds}\right]$. We use equations (8), (9), and (12) and write $\frac{d\tilde{m}}{ds} = -F_\alpha \cdot \left[\lambda_s + \lambda_t \frac{d\tilde{t}}{ds}\right] = -F_\alpha \cdot \frac{d\tilde{\lambda}}{ds}$. Hence, $\left[1 + \lambda_t \cdot \frac{\partial \hat{t}}{\partial m} \cdot F_\alpha\right] \frac{d\tilde{\lambda}}{ds} = \lambda_s + \lambda_t \cdot \frac{\partial \hat{t}}{\partial s}$. Because of Assumption 1 and Lemma 2, we know that $\left[1 + \lambda_t \cdot \frac{\partial \hat{t}}{\partial m} \cdot F_\alpha\right] > 0$ and $\lambda_s + \lambda_t \cdot \frac{\partial \hat{t}}{\partial s} < 0$. We conclude that $\frac{d\tilde{\lambda}}{ds} < 0$ for each s. ■

Proof of Proposition D. We only prove (1) as the proof of (2) follows the same reasoning. We want to prove that $\frac{d\tilde{m}(s)\cdot\tilde{\lambda}(s)}{ds} \geq 0$ for each s if $F - F_\alpha \cdot \lambda \leq 0$ for each pair (s,t). Note that $\frac{d\tilde{m}\tilde{\lambda}}{ds} = \frac{d\tilde{m}}{ds}\tilde{\lambda} + \tilde{m}\frac{d\tilde{\lambda}}{ds}$. From the proof of Proposition C, we know that $\frac{d\tilde{m}}{ds} = -F_\alpha \cdot \frac{d\tilde{\lambda}}{ds}$. Hence, we write $\frac{d\tilde{m}\tilde{\lambda}}{ds} = \frac{d\tilde{\lambda}}{ds}\left[F - F_\alpha \cdot \tilde{\lambda}\right]$. Since $\frac{d\tilde{\lambda}}{ds} < 0$ (Proposition C), we know that the sign of $\frac{d\tilde{m}\tilde{\lambda}}{ds}$ depends on the sign of $\left[F - F_\alpha \cdot \tilde{\lambda}\right]$. We conclude that if $F - F_\alpha \cdot \lambda \leq 0$ for each pair (s,t), then $\frac{d\tilde{m}\tilde{\lambda}}{ds} \geq 0$ for each s. ■

Proof of Proposition F. Let $\tilde{U}_1(s) := U_1(s, \tilde{t}(s), \tilde{m}(s))$, $\tilde{U}_2(s) := U_2(s, \tilde{t}(s), \tilde{m}(s))$, and $\tilde{U}_{TW}(s) := \tilde{U}_1(s) + \tilde{U}_2(s)$. We want to prove $\frac{d\tilde{U}_{TW}}{ds} \geq 0$ for each s. First, we compute $\frac{d\tilde{U}_1}{ds} = \frac{d\tilde{m}}{ds} \cdot \left[1 - \tilde{\lambda}\right] - \tilde{m} \cdot \frac{d\tilde{\lambda}}{ds}$. From the proof of Proposition C, we know that $\frac{d\tilde{m}}{ds} = -F_\alpha \cdot \frac{d\tilde{\lambda}}{ds}$. Hence, we write $\frac{d\tilde{U}_1}{ds} = -F_\alpha \cdot \frac{d\tilde{\lambda}}{ds} \cdot \left[1 - \tilde{\lambda}\right] - F \cdot \frac{d\tilde{\lambda}}{ds}$. Second, using the Leibniz rule, we compute $\frac{d\tilde{U}_2}{ds} = F_\alpha \cdot \frac{d\tilde{\lambda}}{ds} \cdot \left[1 - \tilde{\lambda}\right]$. Consequently, $\frac{d\tilde{U}_{TW}}{ds} = -F \cdot \frac{d\tilde{\lambda}}{ds}$. Since $\frac{d\tilde{\lambda}}{ds} < 0$ (Proposition C), we conclude that $\frac{d\tilde{U}_{TW}}{ds} \geq 0$ for each s. ■