Rationalizing Epistemic Bounded Rationality*

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2015

Abstract

The standard model of knowledge, (Ω, P), consists of state space, Ω, and possibility correspondence, P. Usually, it is assumed that P satisfies all knowledge axioms (Truth Axiom, Positive Introspection Axiom, and Negative Introspection Axiom). Violating at least one of these axioms is defined as epistemic bounded rationality (EBR). If this happens, a researcher may try to look for another model, (Ω^*, P^*), that generates the initial model, (Ω, P), while satisfying all knowledge axioms. Rationalizing EBR means that the researcher finds such a model. I determine when rationalization of EBR is possible. I also investigate when a model, (Ω^*, P^*), which satisfies all knowledge axioms, generates a model, (Ω, P), which satisfies these axioms as well.

Keywords: Knowledge; Epistemic Bounded Rationality

*I wish to thank the anonymous referee, Andrei Gomberg, Joyce Sadka, Tridib Sharma, and the participants of the FUR XIV International Conference for their valuable comments. Support from the Asociación Mexicana de Cultura A.C. is gratefully acknowledged.

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1 Introduction

As stated in Aumann (1976), the classical model of knowledge consists of state space, Ω, and possibility correspondence, P. Each state, ω, is a description of some possible world. Possibility correspondence is a function that assigns a subset of Ω to each ω. $P(\omega)$ is interpreted as a collection of all states that the agent perceives to be possible if ω is a true state. Dekel and Gul (1997), Fagin et al. (1995), Geanakoplos (1989), and Rubinstein (1998) discuss the model in great detail, including its interpretations and proofs of classical results. Knowledge axioms describe the properties of P as follows: Truth Axiom ($\omega \in P(\omega)$), Positive Introspection Axiom (if $\tilde{\omega} \in P(\omega)$, then $P(\tilde{\omega}) \subset P(\omega)$), and Negative Introspection Axiom (if $\tilde{\omega} \in P(\omega)$, then $P(\omega) \subset P(\tilde{\omega})$). Satisfying all knowledge axioms means that the agent is epistemically rational and her possibility correspondence is partitional. Epistemic bounded rationality (EBR) occurs when the agent violates at least one of knowledge axioms (see Geanakoplos (1989) and Rubinstein (1998)).

In this paper, I consider the researcher who builds a model, (Ω, P), that depicts the agent's knowledge. The model hypothesized by the researcher may incorrectly represent the agent's knowledge. In such a case, the researcher may falsely conclude the agent is not epistemically rational. If the researcher detects EBR, then he may suspect that the conclusion derives from his incorrect model of the agent's knowledge. The researcher could investigate whether or not another model, (Ω^*, P^*) exists, which satisfies all knowledge axioms. If such a model does exist, then the researcher rationalizes EBR. In this paper, I consider a specific structure of (Ω^*, P^*). In particular, I impose that Ω^* is defined as $\Omega \times X$ and P in (Ω, P) is a projection of P^* to Ω. Because of its construction, I call (Ω^*, P^*) an extended model of knowledge. In the literature on unawareness, the agent is unaware of some events, but the researcher is assumed to be omnipotent. In this paper, I allow for the researcher who considers only Ω and is unaware of X. The researcher's assumed model, (Ω, P), is derived from (Ω^*, P^*). In Section 2, I discuss the relationship between (Ω^*, P^*) and (Ω, P) and show how the latter is obtained from the former. I also provide an interpretation of X, which expands Ω to Ω^*. In Section 3, I determine when it is possible to rationalize EBR. I do this by fixing a model, (Ω, P), that depicts EBR and investigating when that model can be extended to a model, (Ω^*, P^*), that satisfies all knowledge axioms. In Section 4, I address a complementary problem. I fix a model, (Ω^*, P^*), that satisfies all knowledge axioms and determine when that model generates a model, (Ω, P), that also satisfies all knowledge axioms.

2 Extended model of knowledge

Consider an extended model of knowledge, (Ω^*, P^*), where $\Omega^* = \Omega \times X$. If the researcher hypothesizes that Ω is a state space, then I assume that P is built as a projection of P^* on Ω. This construction is stipulated by the projection-based approach developed in the literature on unawareness (see Board et al. (2011), Galanis (2011), Galanis (2013), Halpern (2001), Halpern and Rêgo (2008), Heifetz et al. (2006), Heifetz et al. (2008), Karni and Vierø (2013), Li (2009), Modica and Rustichini (1999), and Schipper (2013)). First, take a state (ω, x) from Ω^* and let $\underset{\Omega}{\text{proj}} P^*(\omega, x)$ denote the projection of P^* on Ω. That is, $\underset{\Omega}{\text{proj}} P^*(\omega, x)$ is a collection of all states in Ω, $\tilde{\omega}$, such that $(\tilde{\omega}, x)$ is considered to be a possible state when (ω, x) is a true state.

$$\underset{\Omega}{\text{proj}} P^*(\omega, x) := \{\tilde{\omega} \in \Omega : (\tilde{\omega}, x) \in P(\omega, x)\} \tag{1}$$

Second, take all states in Ω^* such that the first coordinate of these states is ω. For each such state, the projection of P^* is then computed. The union of these projections defines the value of P at ω.

$$P(\omega) := \bigcup_{x \in X} \underset{\Omega}{\text{proj}} P^*(\omega, x) \tag{2}$$

Definition 2.1

Fix (Ω^*, P^*) with $\Omega^* = \Omega \times X$. P is a projection of P^* on Ω if P is defined as it is in (2). This P is a possibility correspondence generated by P^*.

The construction of P in Definition 2.1 captures how the researcher, who is unaware of the fact that the agent's state space is $\Omega^* = \Omega \times X$ rather than Ω, determines the agent's possibility correspondence, P over Ω.

The researcher's process of constructing P requires that for each ω in Ω, the researcher obtain an answer to the question: What states in Ω does the agent consider to be possible if ω is a true state? However, in terms of the state space Ω^*, the researcher's question is not precise enough. This is because for each x in X, there is a state (ω, x). Consequently, ω is not a state. Rather, ω is an event denoted by $E_\omega = \{(\omega, x) : x \in X\}$. For every state (ω, x), the projection of $P^*(\omega, x)$ onto Ω is what the agent considers to be possible (in terms of Ω) if (ω, x) is a true state. Consequently, there exists a collection of subsets of Ω generated by taking the projection of $P^*(\omega, x)$ for each x. Each $\underset{\Omega}{\text{proj}} P^*(\omega, x)$ can be interpreted as a conditional answer to the researcher's question. Here, *conditional* refers to the fact that $\underset{\Omega}{\text{proj}} P^*(\omega, x)$ depends on x. However, in this paper, I assume that the researcher obtains an unconditional answer to his question. As explained below, this is because he controls what happens in Ω without specifying what state in X is true (which he can not do because he is unaware of X). The unconditional answer is a combination of all possible conditional answers in the form of a union.

The following example explains the construction of P. Suppose that $\Omega = \{a, b, c\}$ and $X = \{A, B, C\}$. The agent's possibility correspondence satisfies all knowledge axioms and is depicted in Figure 1.

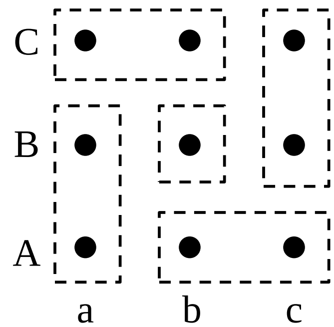

Figure 1

I assume that the researcher's data set is rich enough. That is, for each state ω in Ω, the researcher can determine $P(\omega)$. How is this possible? First, I assume that the agent behaves in accordance with the Subjective Expected Utility theory (Savage (1972)). Second, I assume that the researcher is able to determine what state obtains in a way that prevents the agent from learning what the true state is. Once the researcher fixes a state ω, then he tests which states the agent believes to be Savage-null. The states which are not Savage-null are those that the agent considers to be possible, and these states constitute $P(\omega)$. This way of constructing P requires that state space Ω be finite since no researcher could conduct an infinite number of experiments. In addition, this approach makes more sense in a laboratory setting in which the researcher either has direct power over which state obtains, or conducts an experiment that simulates an occurrence of a specific state. Outside of the laboratory, the researcher generally has no control over what state obtains. However, if the researcher does not choose which state obtains, then it is not clear how he is to construct the agent's possibility correspondence. For example, if the state space is $\{\alpha, \beta\}$, where α stands for "party α wins elections" and β for "party β wins elections," then only one of these two states will obtain but the researcher cannot choose which one. Moreover, if α obtains, then it is not clear how the researcher is to construct $P(\beta)$.

In the example with $\Omega = \{a, b, c\}$, the researcher begins with a state a. In order to construct $P(a)$, he then determines which states in Ω are Savage-null. However, from the perspective of the agent's state space Ω^*, a is not a state but rather an event $\{(a, A), (a, B), (a, C)\}$. I will now analyze each state in that event. First, consider the state (a, A). From the perspective of state space Ω^*, the agent considers both (a, A) and (a, B) to be possible states. However, the relevant perspective is that of state space Ω. From that perspective, the only state that is not Savage-null is a. Note that $\{a\}$ is the projection of $P^*(a, A)$ on Ω. Second, consider the state (a, B). From the perspective of state space Ω, the agent again believes that only state a is possible. Note that $\{a\}$ is the projection of $P^*(a, B)$ on Ω. Finally, consider the state (a, C). Now, in terms of the state space Ω^*, the agent believes that both (a, C) and (b, C) are possible. Consequently, in terms of the state space Ω, only c is Savage-null which means that the agent believes only a and b to be possible. Note that $\{a, b\}$ is the projection of $P^*(a, C)$ on Ω.

To conclude, if the researcher fixes a, then (in terms of subsets of Ω) the agent believes that one of the following is possible: $\{a\}$, or $\{a\}$, or $\{a, b\}$. Formally, this means that $P(a) = \{a\} \cup \{a\} \cup \{a, b\}$, which is the union of projections: $\underset{\Omega}{\text{proj}} P^*(a, A) \cup \underset{\Omega}{\text{proj}} P^*(a, B) \cup \underset{\Omega}{\text{proj}} P^*(a, C)$. This reasoning leads to the construction of P as presented in Definition 2.1.

In the example under consideration, the researcher constructs the agent's possibility correspondence as follows: $P(a) = \{a, b\}$, $P(b) = \{a, b, c\}$, and $P(c) = \{b, c\}$. The researcher detects EBR, however, this conclusion is due to the researcher's misrepresentation of the agent's knowledge.

3 Rationalizable epistemic bounded rationality

Fix a product, $\Omega \times X$, with a possibility correspondence P^* that satisfies all knowledge axioms. In this section, I focus on characterizing the possibility correspondences on Ω which represent EBR and can be obtained from such P^*. Lemma 3.1 identifies the minimal conditions that a possibility correspondence, P, must satisfy in order to be rationalizable. It turns out that the key property is symmetry. Symmetry means that if the agent considers ω_1 to be possible at ω_2 (i.e., $\omega_1 \in P(\omega_2)$), then at ω_1 she believes that ω_2 is possible (i.e., $\omega_2 \in P(\omega_1)$). The symmetry of P is closely related to the B(rouwerian) Axiom, known from the modal logic literature (see Chellas (1995), Hughes and Cresswell (1984), and Hughes and Cresswell (1996)). If the agent's possibility correspondence is symmetric, then the agent knows that she considers an event E to be possible whenever E obtains (that is, whenever a true state is a member of E). If the event does not obtain, then the agent knows that she does not know E. I also show that violation of the Truth Axiom is not rationalizable.

Lemma 3.1

Fix (Ω^*, P^*) with $\Omega^* = \Omega \times X$. Let P be a possibility correspondence on Ω that is generated by P^*.

1. *If P^* satisfies all knowledge axioms, then P is symmetric.*
2. *If P^* satisfies the Truth Axiom, then P satisfies the Truth Axiom.*

Proof of Lemma 3.1: 1. Suppose that $\omega_1 \in P(\omega_2)$. By definition of P, there must be x_1 and x_2 such that $(\omega_1, x_1) \in P^*(\omega_2, x_2)$. Hence, $P^*(\omega_1, x_1) = P^*(\omega_2, x_2)$. Consequently, $(\omega_2, x_2) \in P^*(\omega_1, x_1)$, and again, by definition of P, $\omega_2 \in P(\omega_1)$. 2. Take $\omega \in \Omega$. By assumption, $(\omega, x) \in P^*(\omega, x)$ for each $x \in X$. Hence, by construction of P, $\omega \in P(\omega)$. ■

If a symmetric P satisfies the Positive Introspection Axiom, then it must be true that it does not violate the Negative Introspection Axiom. To confirm that this is true, assume that such a P violates the Negative Introspection Axiom. Hence, there are states ω_1 and ω_2 such that $\omega_1 \in P(\omega_2)$ and $P(\omega_2)$ is not a subset of $P(\omega_1)$. By symmetry, $\omega_2 \in P(\omega_1)$, but by the Positive Introspection Axiom, $P(\omega_2) \subset P(\omega_1)$. This is a contradiction. Similarly, a model that satisfies the Negative Introspection Axiom, but violates the Positive Introspection Axiom also cannot be rationalized. As Lemma 3.1 indicates, each model that can possibly be rationalized must satisfy the Truth Axiom. However, it is well known that the Truth Axiom and the Negative Introspection Axiom imply the Positive Introspection Axiom.

Proposition 1

Suppose that P on Ω is symmetric and satisfies the Truth Axiom. Fix X such that X and Ω have the same cardinality. Then, there exists P^*, defined on $\Omega \times X$, such that P^* satisfies all knowledge axioms and generates P.

Proof of Proposition 1: Since X and Ω have the same cardinality, there exists a bijection $f : \Omega \to X$. For each ω, I define cylinder $C_\omega = \{\omega\} \times X$. I partition C_ω in such a way that, for each $\hat{\omega} \in P(\omega)$, there exists a nonempty subset of C_ω. I denote this subset by $B_\omega(\hat{\omega})$. Since $P(\omega)$ is a subset of Ω and X has the same cardinality as Ω, the proposed construction is possible. I also require that a state not in $P(\omega)$ not be assigned a subset of C_ω. That is,

$\bigcup_{\hat{\omega}\in P(\omega)} B_\omega(\hat{\omega}) = C_\omega$. In other words, $B_\omega(\hat{\omega})$ is nonempty if and only if $\hat{\omega} \in P(\omega)$. I construct P^* on $\Omega \times X$ using the sets $B_\omega(\hat{\omega})$s in the following way. First, each individual $B_\omega(\omega)$ is part of the partition P^* on its own. That is, $P^*(\omega_1, x) = B_\omega(\omega)$ for all $(\omega_1, x) \in B_\omega(\omega)$. This leaves the sets $B_\omega(\hat{\omega})$s for $\hat{\omega} \neq \omega$. Take this nonempty $B_\omega(\hat{\omega})$. Due to symmetry, there exists a nonempty $B_{\hat{\omega}}(\omega)$. I create the union of these two, which then becomes part of the partition P^*. That is, $P^*(\omega_1, x) = B_\omega(\hat{\omega}) \cup B_{\hat{\omega}}(\omega)$ for all $(\omega_1, x) \in B_\omega(\hat{\omega}) \cup B_{\hat{\omega}}(\omega)$. In order to verify that the proposed construction yields the desired result, first note that P^* satisfies all of the knowledge axioms. Second, I show that the possibility correspondence over Ω, $\bar{P}$, which is generated by P^*, equals P. Fix ω and let $\hat{\omega} \in \bar{P}(\omega)$. By definition, $\hat{\omega} \in \bar{P}(\omega)$ if and only if there are $\tilde{\omega}$ and $\tilde{\tilde{\omega}}$ such that $(\hat{\omega}, f(\tilde{\omega})) \in P^*(\omega, f(\tilde{\tilde{\omega}}))$. By construction, P^* is the union of B subsets. These different first coordinates preclude $(\hat{\omega}, f(\tilde{\omega}))$ from being a member of any $B_\omega(.)$. Thus, it must be true that $(\hat{\omega}, f(\tilde{\omega})) \in B_{\hat{\omega}}(\bar{\omega})$ for some $\bar{\omega}$. Hence, by construction, $P^*(\omega, f(\tilde{\tilde{\omega}})) = B_{\hat{\omega}}(\bar{\omega}) \cup B_{\bar{\omega}}(\hat{\omega})$. However, note that $\bar{\omega} = \omega$ because $(\omega, f(\tilde{\tilde{\omega}})) \in B_{\hat{\omega}}(\bar{\omega}) \cup B_{\bar{\omega}}(\hat{\omega})$, which would not be possible otherwise because of the differences in the first coordinates. Thus, it is possible to conclude that $B_\omega(\hat{\omega})$ is nonempty. However, as argued above, this is true if and only if $\hat{\omega} \in P(\omega)$. ■

The proposed structure of the extended state space Ω^* consists of the state space Ω assumed by the researcher and an extension X. That is, $\Omega^* = \Omega \times X$. It is possible to interpret $\Omega^* = \Omega \times X$ in two ways.

The dynamic interpretation indicates that X captures additional periods of which the researcher is unaware. Proposition 1 requires that X has the same cardinality as Ω. Consequently, if the researcher starts with a model (Ω, P) that describes EBR, and it is possible to rationalize EBR, then it is enough to consider $\Omega^* = \Omega \times \Omega$. In other words, extending the state space to include more periods provides no additional gain.

The second interpretation of $\Omega^* = \Omega \times X$ is called the descriptive interpretation. Set X may consist of the missing description of the agent's state space. In that case, the assumed Ω is incomplete because the researcher did not take into account all of the aspects the agent is considering. This interpretation is particularly useful if the state space is built in a canonical way (see chapter 3 in Gilboa (2009)). Consider the following example. From the agent's perspective, there are two relevant elements involved in describing the world, s and h. Symbol $\neg$ indicates negation. Hence, $\{(s, h), (s, \neg h), (\neg s, h), (\neg s, \neg h)\}$ is an example of a state space. However, if the researcher assumes that only s is a relevant aspect, then he would build only two states of the world, $\Omega = \{(s), (\neg s)\}$. In this case, replacing the researcher's construction by the product $\Omega \times X$, where $X = \{(h), (\neg h)\}$, would recreate the agent's view. Recall, however, that the cardinality of X must be at least equal to the cardinality of Ω.

4 Generating epistemic rationality

In this section, I identify the conditions under which P^* over $\Omega \times X$ always generates epistemic rationality on Ω. As Lemma 3.1 shows, P generated by such a P^* satisfies the Truth Axiom, and the only form of possible EBR is a simultaneous violation of the Positive and Negative Introspection Axioms. Hence, it is necessary to determine when P will satisfy both introspection axioms.

The first obvious candidate is a product of P and P_X, where P and P_X are defined over Ω and X, respectively.

Lemma 4.1

Assume that P^* over $\Omega \times X$ is defined as a product, $P^*(\omega, x) = P(\omega) \times P_X(x)$. Then, P^* satisfies all knowledge axioms if and only if both P and P_X satisfy all knowledge axioms.

Proof of Lemma 4.1: First, assume that P^* satisfies all knowledge axioms. I show that P satisfies all knowledge axioms. The proof for P_X is identical. By Lemma 3.1, P satisfies the Truth Axiom. Take ω_1 and ω_2 such that $\omega_2 \in P(\omega_1)$. Hence, $(\omega_2, x_2) \in P^*(\omega, x_1)$ for some x_1 and x_2, and $P^*(\omega_2, x_2) = P^*(\omega_1, x_1)$. Since P^* is a product, $P^*(\omega_2, x_2) = P(\omega_2) \times P_X(x_2) = P(\omega_1) \times P_X(x_1) = P^*(\omega_1, x_1)$. This implies that $P(\omega_2) = P(\omega_1)$. Next, assume that P and P_X satisfy all knowledge axioms. Take (ω, x). Since both P and P_X satisfy the Truth Axiom, it is true that $(\omega, x) \in P(\omega) \times P_X(x) = P^*(\omega, x)$. In order to prove that P^* satisfies the Positive and Negative Introspection Axioms, take (ω_1, x_1) and assume that $(\omega_2, x_2) \in P^*(\omega_1, x_1)$. Thus, $\omega_2 \in P(\omega_1)$ and $x_2 \in P_X(x_1)$. In consequence, $P(\omega_1) = P(\omega_2)$ and $P_X(x_1) = P_X(x_2)$. That is, $P^*(\omega_1, x_1) = P^*(\omega_2, x_2)$. ■

Another natural candidate is a scenario consisting of a two-element state space, $\Omega = \{\omega_1, \omega_2\}$. Suppose that a possibility correspondence, but not necessarily a product, on $\Omega \times X$ satisfies all knowledge axioms. Then, the only scenario in which P represents EBR is the case of $P(\omega_1) = \Omega$, with $P(\omega_2)$ being a strict subset of Ω. Given Lemma 3.1, it is necessary that $P(\omega_2) = \{\omega_2\}$, in order to have a chance to rationalize P. However, such a possibility correspondence violates only the Negative Introspection Axiom, which means that, as shown in the same lemma, P cannot be rationalized. This implies a contradiction.

In order to discuss the conditions for P^* that guarantee the epistemic rationality of P for $|\Omega| > 2$, I first must introduce a new concept.

Definition 4.1

Fix (Ω^*, P^*) *with* $\Omega^* = \Omega \times X$*. Two states,* ω_1 *and* ω_2 *in* Ω*, are connected if there exist* x_1*,* x_2*,* $\tilde{\omega}$*, and* $\tilde{x}$ *such that* $(\omega_1, x_1), (\omega_2, x_2) \in P^*(\tilde{\omega}, \tilde{x})$*.*

Connectedness is at the core of understanding how the transfer of epistemic rationality from P^* to P works. The following result reveals the key implication of connectedness.

Lemma 4.2

Fix (Ω^*, P^*) *with* $\Omega^* = \Omega \times X$*. Let* P *be a possibility correspondence on* Ω *that is generated by* P^**. Assume that* P^* *satisfies the Truth Axiom. Then,* ω_1 *and* ω_2 *are connected if and only if* $\omega_1 \in P(\omega_2)$*.*

Proof of Lemma 4.2: First suppose that ω_1 and ω_2 are connected. Hence, there are x_1, x_2, $\tilde{\omega}$, and $\tilde{x}$ such that $(\omega_1, x_1), (\omega_2, x_2) \in P^*(\tilde{\omega}, \tilde{x})$. Since P^* satisfies the Truth Axiom, it is true that $P^*(\tilde{\omega}, \tilde{x}) = P^*(\omega_2, x_2)$. Thus, $(\omega_1, x_1) \in P^*(\omega_2, x_2)$, which, by construction of P, implies that $\omega_1 \in P(\omega_2)$. Next assume that $\omega_1 \in P(\omega_2)$. Again by construction of P, this implies that, for some x_1, (ω_1, x_1) must belong to $P^*(\omega_2, x_2)$ for some x_2. Due to the Truth Axiom, $(\omega_2, x_2) \in P^*(\omega_2, x_2)$. But this implies that $(\omega_1, x_1), (\omega_2, x_2) \in P^*(\omega_2, x_2)$, which is a definition of connectedness. ■

I close this paper with the proposition that identifies the necessary and sufficient conditions for P to violate the introspection axioms. Essentially, transitivity of connectedness is required.

Proposition 2

Fix (Ω^*, P^*) *with* $\Omega^* = \Omega \times X$*. Let* P *be a possibility correspondence on* Ω *that is generated by* P^**. Assume that* P^* *satisfies all knowledge axioms. The following are then equivalent:*

1. *There are states* ω_1*,* ω_2*, and* ω_3 *such that: (a)* ω_1 *and* ω_2 *are connected, (b)* ω_2 *and* ω_3 *are connected, and (c)* ω_1 *and* ω_3 *are not connected.*

2. *There are states* ω_1 *and* ω_2 *such that* $P(\omega_1) \cap P(\omega_2) \neq \emptyset$ *and* $P(\omega_1) \neq P(\omega_2)$*.*

Proof of Proposition 2: 1. Assume (1). Since ω_1 and ω_2 are connected, Lemma 4.2 tells that $\omega_1 \in P(\omega_2)$ and $\omega_2 \in P(\omega_1)$. Hence, $P(\omega_1) \cap P(\omega_2) \neq \emptyset$. Similarly, $\omega_3 \in P(\omega_2)$. However, ω_1 and ω_3 are not connected. Lemma 4.2 again implies that $\omega_3 \notin P(\omega_1)$. Thus, $P(\omega_1) \neq P(\omega_2)$. 2. Assume (2). Two cases then need to be considered. First, suppose that ω_1 and ω_2 are connected. As such, (a) is already satisfied. Since $P(\omega_1)$ and $P(\omega_2)$ differ, there must be ω_3 such that either $\omega_3 \in P(\omega_1)$ and $\omega_3 \notin P(\omega_2)$ or $\omega_3 \in P(\omega_2)$ and $\omega_3 \notin P(\omega_1)$. Without loss of generality, I presume that the latter holds. Due to Lemma 4.2, this implies that ω_2 and ω_3 are connected, while ω_1 and ω_3 are disconnected. Hence, both (b) and (c) hold. Second, suppose that ω_1 and ω_2 are disconnected which, after relabeling the states, yields condition (c). Hence, $\omega_1 \notin P(\omega_2)$ and $\omega_2 \notin P(\omega_1)$. Since $P(\omega_1)$ and $P(\omega_2)$ are not disjoint, there must be ω_3 such that $\omega_3 \in P(\omega_1)$ and $\omega_3 \in P(\omega_2)$. By Lemma 4.2, ω_1 and ω_3 are connected, and ω_2 and ω_3 are connected. After renaming the states, this yields conditions (a) and (b). ■

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